

Vector Algebra

Vector notation is designed to not just give you a point in space but a directional vector. They will usually look like $\langle 2,5 \rangle$. This gives you a vector that starts at the origin and continues in the direction of the (x, y) point of (2,5). Vectors can also be written in the form: $2\mathbf{i} + 5\mathbf{j}$. The i component is always x and the j is y.

Basic Vector Operations and Rules (let $\mathbf{u} = \langle a,b \rangle$ and $\mathbf{v} = \langle c,d \rangle$)

Magnitude of a vector: $|\mathbf{v}| = \sqrt{a^2 + b^2}$; length of the vector

Addition Rules: $\mathbf{u} + \mathbf{v} = \langle a,b \rangle + \langle c,d \rangle = \langle a+c, b+d \rangle$

Scalar Multiplication: $c\mathbf{u} = c\langle a,b \rangle = \langle ca, cb \rangle$

Examples of addition and subtraction with vectors, scalar multiplication:

Let: $\mathbf{U} = \langle 2,3 \rangle$, $\mathbf{W} = \langle -4, 6 \rangle$

- 1) $\mathbf{U} + \mathbf{W} = \langle 2-4, 3+6 \rangle = \langle -2, 9 \rangle$ or $-2\mathbf{i} + 9\mathbf{j}$
- 2) $\mathbf{W} - \mathbf{U} = \langle -4, 6 \rangle - \langle 2,3 \rangle = \langle -4-2, 6-3 \rangle = \langle -6, 3 \rangle$ or $-6\mathbf{i} + 3\mathbf{j}$
- 3) $2\mathbf{U} - 3\mathbf{W} = \langle 4, 6 \rangle - \langle -12, 18 \rangle = \langle 4 - (-12), 6 - 18 \rangle = \langle 16, -12 \rangle$ or $16\mathbf{i} - 12\mathbf{j}$

You Try (1): find $4\mathbf{U} - 2\mathbf{W}$

Unit Vector- magnitude is 1

Example of converting a vector to a unit vector: $\mathbf{v} = \langle 2,5 \rangle$

- 1) Find the magnitude $\rightarrow \sqrt{2^2 + 5^2} = \sqrt{4 + 25} = \sqrt{29}$
- 2) $\mathbf{U} = \frac{1}{|\mathbf{v}|} \mathbf{v} = \langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \rangle$
- 3) This would give a unit vector (length of 1) in the direction of \mathbf{v} .

You Try(2): find the unit vector for the vector $\mathbf{v} = \langle -1, 5 \rangle$

Dot Product

The dot product is used to find the angle between two vectors. The result will always be a scalar (a number not a vector).

Definition: if $\mathbf{u} = \langle a,b \rangle$ and $\mathbf{v} = \langle c,d \rangle$ then $\mathbf{u} \cdot \mathbf{v} = ac + bd$

Example: $\mathbf{u} = \langle 2,3 \rangle$, $\mathbf{v} = \langle -4, 6 \rangle$, then $\mathbf{u} \cdot \mathbf{v} = 2 \times -4 + 3 \times 6 = -8 + 18 = 10$

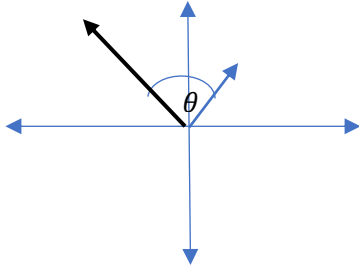
You Try(3): $\mathbf{u} = \langle 1, 5 \rangle$, $\mathbf{v} = \langle -2, 1 \rangle$ solve for $\mathbf{u} \cdot \mathbf{v}$

Find the angle between two vectors (application of the dot product)

$$\cos \theta = \frac{u \cdot v}{|u||v|}$$

So if $U = \langle 2,3 \rangle$, $V = \langle -4, 6 \rangle$, then:

$$\cos \theta = \frac{\langle 2,3 \rangle \cdot \langle -4,6 \rangle}{|\sqrt{13}||\sqrt{52}|} = \frac{-8+18}{26} = \frac{10}{26} = \frac{5}{13}; \text{ thus } \cos \theta = \frac{5}{13}; \text{ so } \theta = 67.38^\circ.$$

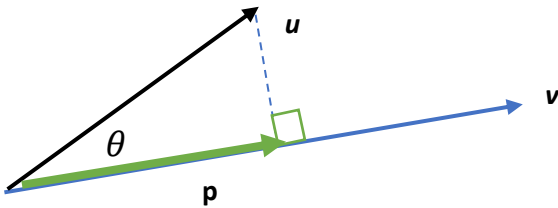


On another note, two vectors are said to “orthogonal” (at a right angle or 90° apart) if $u \cdot v = 0$

You Try(4): find the angle between $U = \langle -2,1 \rangle$ and $V = \langle -4, 3 \rangle$

Finding the scalar component of u on v

An important use of the dot product is to determine the projection of one vector onto another if they share a common initial point.



If θ is the angle between vectors u and v , then the scalar component of u on v is given by:

$$\text{comp}_v u = \frac{u \cdot v}{|v|}$$

So if $u = \langle 2,3 \rangle$, $v = \langle -4, 6 \rangle$, then

$$\text{comp}_v u = \frac{\langle 2,3 \rangle \cdot \langle -4,6 \rangle}{|\sqrt{13}|} = \frac{10}{\sqrt{13}} = 2.77 \text{ to 2 significant figures.}$$

One application of this is Work which can be defined as:

$$\text{comp}_d F |d| = \frac{F \cdot d}{|d|} |d| = F \cdot d$$

Answers to You Try's: 1) $\langle 16, 0 \rangle$ 2) $U = \langle \frac{-1}{\sqrt{26}}, \frac{5}{\sqrt{26}} \rangle$ 3) 3 4) 26.57°