

Logarithms

A **logarithm** of a given number x , is the exponent required for the base a , to be raised to in order to produce that number x .

$$\log_a x = y \quad \Leftrightarrow \quad a^y = x$$

Note that \Leftrightarrow means "is equivalent to"

Logarithmic and Exponential Form

Change logarithm equations to exponential form or exponential equations to logarithmic form using the definition of a logarithm.

Example: Given $4^{3/2} = 8$, change the equation to logarithmic form.

Solution:

Compare the equation to the definition and rewrite it.

Definition: $\log_a x = y \Leftrightarrow a^y = x$ \longleftrightarrow Notice that $a = 4$, $x = 8$, and
 Given: $4^{3/2} = 8$ $y = \frac{3}{2}$, respectively.

Therefore, using the definition: $4^{3/2} = 8 \Leftrightarrow \log_4 8 = \frac{3}{2}$

Example: Given $\log_{25} 5 = \frac{1}{2}$, change the equation to exponential form.

Solution:

Compare the equation to the definition and rewrite it.

Definition: $\log_a x = y \Leftrightarrow a^y = x$ \longleftrightarrow Notice that $a = 25$, $x = 5$, and
 Given: $\log_{25} 5 = \frac{1}{2}$ $y = \frac{1}{2}$, respectively.

Therefore, using the definition: $\log_{25} 5 = \frac{1}{2} \Leftrightarrow 25^{1/2} = 5$

Solving Logarithm and Exponential Equations

Evaluate logarithmic equations by using the definition of a logarithm to change the equation into a form that can then be solved.

Example: Given $3^{x-1} = 7$, solve for x .

Solution:

Step 1: Set up the equation and use the definition to change it.

Definition: $\log_a x = y \Leftrightarrow a^y = x$
Given $3^{x-1} = 7$

Notice 3 is the base or a ,
and 7 is the given number.

$$3^{x-1} = 7 \Leftrightarrow \log_3 7 = x - 1$$

Step 3: Use the order of operations to finish solving for x .

$$x - 1 = \frac{\log 7}{\log 3}$$

$$x = \frac{\log 7}{\log 3} + 1$$

Example: Given $\log_6(x + 2) = 3$, solve for x .

Solution:

Step 1: Set up the equation and use the definition to change it.

Definition: $\log_a x = y \Leftrightarrow a^y = x$
Given $\log_6(x + 2) = 3$
Notice 6 is the base or a ,
and 3 is the exponent or y .

$$\log_6(x + 2) = 3 \Leftrightarrow 6^3 = x + 2$$

Step 2: Now use the properties of logarithms to solve.

Recall the Change of Base Property:
 $\log_a b = \frac{\log b}{\log a}$

Apply it to $\log_3 7$.

$$\log_3 7 = \frac{\log 7}{\log 3}$$



Step 2: Now use the order of operations to solve.

$$6^3 = x + 2$$

$$216 = x + 2$$

$$214 = x$$

$$x = \mathbf{214}$$

Expanding and Simplifying Logarithms

To expand or simplify logarithms, utilize the various properties of logarithms in conjunction with the definition.

Example: Given $\log_3 \left(\frac{9x^2}{\sqrt{x^2 + 1}} \right)$, expand the logarithm.

Solution:

Step 1: Expand the expression using the properties of logarithms.

Recall the Logarithm Multiplication and Division Properties:

$$\log_a mn = \log_a m + \log_a n$$

$$\log_a \left(\frac{m}{n} \right) = \log_a m - \log_a n$$

Apply them to $9x^2$ and $\sqrt{x^2 + 1}$.

$$\text{Given } \log_3 \left(\frac{9x^2}{\sqrt{x^2 + 1}} \right):$$

$$\Rightarrow \log_3 9 + \log_3 x^2 - \log_3 (\sqrt{x^2 + 1})$$

Step 2: Now simplify further using the properties of logarithms and the definition.

Recall the Logarithm for Powers Property:

$$\log_a x^c = c \log_a x$$

Apply it to the x^2 and $\log_3 (\sqrt{x^2 + 1})$.

$$\begin{aligned} & \log_3 9 + \log_3 x^2 - \log_3 (\sqrt{x^2 + 1}) \\ & \Rightarrow \log_3 9 + \log_3 x^2 - \log_3 (x^2 + 1)^{1/2} \\ & \Rightarrow \log_3 9 + 2 \log_3 x - \frac{1}{2} \log_3 (x^2 + 1) \end{aligned}$$

By definition, $\log_3 9 = 2$ since $3^2 = 9$, so our final answer becomes:

$$\mathbf{2 + 2 \log_3 x - \frac{1}{2} \log_3 (x^2 + 1)}$$

Example: Write $3 \log_2 y - \log_2 x - 7 \log_2 z$ as a single logarithm.

Solution:

To simplify the expression, work backwards with the logarithmic properties.

Step 1: Use the Logarithm for Powers Property where appropriate.

Given: $3 \log_2 y - \log_2 x - 7 \log_2 z$
Notice that it can be applied to $3 \log_2 y$ and $7 \log_2 z$.

$$\begin{aligned} & 3 \log_2 y - \log_2 x - 7 \log_2 z \\ & \Rightarrow \log_2 y^3 - \log_2 x - \log_2 z^7 \end{aligned}$$

Step 2: Simplify using the Logarithm Multiplication and Division Properties. Use the order of operations as a guide.

$$\begin{aligned} & \log_2 y^3 - \log_2 x - \log_2 z^7 \\ & \Rightarrow \log_2 y^3 - (\log_2 x + \log_2 z^7) \\ & \Rightarrow \log_2 y^3 - \log_2 xz^7 \\ & \Rightarrow \mathbf{\log_2 \frac{y^3}{xz^7}} \end{aligned}$$

Solving Expanded Logarithms

Solving expanded logarithms requires applying the definition of logarithms and all the logarithm properties as needed.

Example: Given $\ln(x - 2) + \ln(x - 3) = \ln(2x + 24)$, solve for x .

Solution:

Note: $\ln(x - 2)$ is only valid if $x \geq 2$, $\ln(x - 3)$ is only valid if $x \geq 3$, and $\ln(2x + 24)$ is only valid if $x \geq -12$. For the equation to be valid, all conditions must be met, so $x \geq 3$.

Step 1: Simplify the left side of the equation using the multiplication and division properties of logarithms.

$$\ln(x - 2) + \ln(x - 3) = \ln(2x + 24)$$

$$\begin{aligned}\Rightarrow \ln(x - 2)(x - 3) &= \ln(2x + 24) \\ \Rightarrow \ln(x^2 - 5x + 6) &= \ln(2x + 24)\end{aligned}$$

Step 2: Use logarithm properties.

Recall logarithm properties of bases:
 $\ln e^x = x$ and $e^{\ln x} = x$

$\ln(x^2 - 5x + 6) = \ln(2x + 24)$
Let both sides of the equation become the exponent of the base e , and apply the property.

$$\begin{aligned}\Rightarrow e^{\ln(x^2 - 5x + 6)} &= e^{\ln(2x + 24)} \\ \Rightarrow x^2 - 5x + 6 &= 2x + 24\end{aligned}$$

Practice Exercises:

- Given $\log_4(-x) + \log_4(6 - x) = 2$, Solve for x .
- Expand $\log_2\left(\frac{x}{\sqrt{x^2 - 1}}\right)$ completely.
- Write the following as a single logarithm: $2 \log_3 x + 4 - 8 \log_3 y$

Step 3: Combine like terms to solve for x .

$$\begin{aligned}x^2 - 5x + 6 &= 2x + 24 \\ \Rightarrow x^2 - 7x - 18 &= 0 \\ \Rightarrow (x - 9)(x + 2) &= 0 \\ x &= 9, -2\end{aligned}$$

Step 4: Check your answers. Recall that every logarithm must meet the conditions for the answer to be correct.

For $x = 9$
 $\ln((9) - 2) + \ln((9) - 3) = \ln(2(9) + 24)$
 $\Rightarrow \ln(7) + \ln(6) = \ln(42)$
 $\Rightarrow \ln(7 \cdot 6) = \ln(42) \rightarrow$ This is valid!

For $x = -2$
Since $-2 \not\geq 3$, it does not meet all the conditions, and is not valid.

Therefore: $x = 9$

Answers:

- $x = -2$
- $\log_2 x - \frac{1}{2} \log_2(x - 1) - \frac{1}{2} \log_2(x + 1)$
- $\log_3 \frac{81x^3}{y^8}$