Using the Limit Definition to find the Derivative

The Process:

1. Identify $f(x)$.
2. Substitute and find $f(x+h)$.
3. Substitute these into the limit definition (difference quotient)
4. Simplify
5. Substitute h= 0 into the remaining pieces and simplify
6. The result is the derivative of your function.

Sympolic Example:

$f'(x) = \frac{f(x+h)-f(x)}{h}$; where f(x) is differential.

Example:

$f(x) = ax^2 + bx + c$

$$\begin{aligned} f(x+h) &= a(x+h)^2 + b(x+h) + c \\ &= a(x^2 + 2xh + h^2) + bx + bh + c \\ &= ax^2 + 2axh + ah^2 + bx + bh + c \end{aligned}$$

Using the definition- limit of the difference quotient:

$f'(x) = \frac{f(x+h)-f(x)}{h}$; substitute and simplify.

$$\lim_{h\to 0} \frac{[a(x+h)^2 + b(x+h) + c] - (ax^2 + bx + c)}{h}$$

$$\lim_{h\to 0} \frac{ax^2 + 2axh + ah^2 + bx + bh + c - (ax^2 + bx + c)}{h}$$

Distribute and group like terms.

$$\lim_{h\to 0} \frac{(ax^2 - ax^2) + 2axh + ah^2 + (bx - bx) + bh + (c - c)}{h}$$

$$\lim_{h\to 0} \frac{\cancel{(ax^2 - ax^2)} + 2axh + ah^2 + \cancel{(bx - bx)} + bh + \cancel{(c - c)}}{h}$$

$\lim_{h\to 0} \frac{2axh + ah^2 + bh}{h}$ ***Note****: only terms with an h should remain.*

Reduce the fraction before applying the limit.

$$\lim_{h\to 0} \frac{\cancel{h}(2ax + ah + b)}{\cancel{h}}$$

Apply the limit

$$\lim_{h\to 0} (2ax + ah + b) = 2ax + a(0) + b$$

Thus, $\boldsymbol{f'(x) = 2ax + b}$

Quadratic Example:

Given: $f(x) = 5x^2 + 3x + 12$, *what is* $f'(x)$?

$f(x) = 5x^2 + 3x + 12$,

$$\begin{aligned} f(x+h) &= 5(x+h)^2 + 3(x+h) + 12 \\ &= 5(x^2 + 2xh + h^2) + 3x + 3h + 12 \\ &= 5x^2 + 10xh + 5h^2 + 3x + 3h + 12 \end{aligned}$$

Using the definition- difference quotient:

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$; substitute and simplify.

$$\lim_{h\to 0} \frac{5x^2 + 10xh + 5h^2 + 3x + 3h + 12 - (5x^2 + 3x + 12)}{h}$$

Distribute and group like terms

$$\lim_{h\to 0} \frac{5x^2 + 10xh + 5h^2 + 3x + 3h + 12 - 5x^2 - 3x - 12}{h}$$

$$\lim_{h\to 0} \frac{(5x^2 - 5x^2) + 10xh + 5h^2 + (3x - 3x) + 3h + (12 - 12)}{h}$$

$$\lim_{h\to 0} \frac{\cancel{(5x^2 - 5x^2)} + 10xh + 5h^2 + \cancel{(3x - 3x)} + 3h + \cancel{(12 - 12)}}{h}$$

$$\lim_{h\to 0} \frac{10xh + 5h^2 + 3h}{h}$$

Reduce the fraction before applying limit

$$\lim_{h\to 0} \frac{10x\cancel{h}}{\cancel{h}} + \frac{5h^{\cancel{2}}}{\cancel{h}} + \frac{3\cancel{h}}{\cancel{h}}$$

$$\lim_{h\to 0} (10x + 5h + 3)$$

Apply the limit

$$\lim_{h\to 0} (10x + 5h + 3) = 10x + 5(0) + 3$$

$\boldsymbol{f'(x) = 10x + 3}$

Radical Example (using the conjugate):

$f(x) = \sqrt{x}$

$f(x+h) = \sqrt{x+h}$

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$; by definition

$\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$;

To simplify this, we can use the conjugate.

$\lim_{h\to 0} \frac{(\sqrt{x+h}-\sqrt{x})}{h} * \frac{(\sqrt{x+h}+\sqrt{x})}{(\sqrt{x+h}+\sqrt{x})}$

***Foil out the top and simplify it. Do not foil out the bottom. Keep the components separate.**

$\lim_{h\to 0} \frac{\sqrt{x+h}\sqrt{x+h}-\sqrt{x+h}\sqrt{x}+\sqrt{x+h}\sqrt{x}-\sqrt{x}\sqrt{x}}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$

$\lim_{h\to 0} \frac{\sqrt{x+h}\sqrt{x+h}-\cancel{\sqrt{x+h}\sqrt{x}}+\cancel{\sqrt{x+h}\sqrt{x}}-\sqrt{x}\sqrt{x}}{h(\sqrt{x+h}+\sqrt{x})}$

$\lim_{h\to 0} \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}$; ***continue simplifying***

$\lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})}$; ***reduce the h terms***

$\lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h}+\sqrt{x})}$

$\lim_{h\to 0} \frac{1}{(\sqrt{x+h}+\sqrt{x})}$; ***apply the limit***

$\lim_{h\to 0} \frac{1}{(\sqrt{x+0}+\sqrt{x})} = \frac{\mathbf{1}}{\mathbf{2\sqrt{x}}} = \boldsymbol{f'(x)}$

Trigonometry Example

$\text{f(x)} = \sin(\text{x})$

$\text{f(x + h)} = \sin(\text{x + h})$

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$; by definition

$\lim_{h\to 0} \frac{sin(x+h)-sin(x)}{h}$

To simplify this one, we need to remember a few more formulas from the trig identities.

$\boldsymbol{sin(a+b) = sin(a)cos(b) + cos(a)\mathbf{sin}(b)}$

$\boldsymbol{\lim_{x\to 0} \frac{sin\ x}{x} = 1} \qquad \boldsymbol{\lim_{x\to 0} \frac{cos(x)-1}{x} = 0}$

Applying the sum formula, we have:

$\lim_{h\to 0} \frac{\sin(x)\cos(h)+\cos(x)\sin(h) - sin(x)}{h}$

We can do some grouping, with the formulas in mind.

$\lim_{h\to 0} \frac{sin(x)cos(h) - sin(x)}{h} + \lim_{h\to 0} \frac{cos(x)sin(h)}{h}$

$\lim_{h\to 0} \frac{sin(x)[cos(h)-1]}{h} + \lim_{h\to 0} \frac{cos(x)sin(h)}{h}$

$\lim_{h\to 0} sin(x)\left(\frac{cos(h)-1}{h}\right) + \lim_{h\to 0} cos(x)\left(\frac{sin(h)}{h}\right)$

$\lim_{h\to 0} [\sin(x)(0) + \cos(x)(1)]$

$\boldsymbol{f'(x) = \mathbf{cos}\,(x)}$

You Try:

1. $f(x) = 2x + 4$
2. $f(x) = 2x^2 + 5x - 3$
3. $f(x) = \frac{1}{\sqrt{x}}$
4. $f(x) = \cos(x)$

Solutions:

1. $f'(x) = 2$ **2.** $f'(x) = 4x + 5$ 3. $f'(x) = \frac{-1}{2\sqrt{x^3}}$ 4. $f'(x) = -sin(x)$